Topology Qual, Algebraic Topology:
Summer 2012

(1) Let Σ_g denote the closed, orientable, surface of genus g.
Prove that if Σ_g is a covering space of Σ_h, then there is a $d \in \mathbb{Z}^+$ satisfying

$$g = d(h - 1) + 1\,.$$

(2) Let X be a closed (i.e., compact & boundaryless), orientable k-dimensional manifold. Prove that if $H_{k-1}(X;\mathbb{Z})$ is torsion-free, then so is $H_k(X;\mathbb{Z})$.

(3) Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, concretely identified as the quotient space of the Euclidean plane by the standard integer lattice. Then any 2×2 integer matrix A induces a map

$$\phi : (\mathbb{R}/\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2$$

by left (matrix) multiplication.

(a) Show that (with respect to a suitable basis) the induced contravariant map

$$\phi^* : H^1(T^2;\mathbb{Z}) \leftarrow H^1(T^2;\mathbb{Z})$$

on the cellular cohomology is left multiplication by the transpose of A.

(b) Since T^2 is a closed, $\mathbb{Z}$-oriented manifold, it has a fundamental class, $[T^2] \in H_2(T^2;\mathbb{Z})$. Prove that

$$\phi_*[T^2] = \det(A) \cdot [T^2].$$

(Hint: Use part (a) and the naturality of the cup product under induced maps on homology/cohomology.)

(4) The closed, orientable surface Σ_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R (often called a genus g solid handlebody).

Two copies of R, glued together by the identity map between their boundary surfaces, form a closed 3-manifold X. Compute $H_*(X;\mathbb{Z})$.

GT Qual 2012 (Spring) Part II

Show All Relevant Work!

1) Consider stereographic projection of the unit circle S^1 in $\mathbb{R}^2$ to $\mathbb{R}$ from the North Pole (σ) and from the South Pole ($\tilde{\sigma}$).

a) Show that $\tilde{\sigma} \circ \sigma^{-1}(x) = 1/x$

b) Consider the smooth vector field $\frac{d}{dx}$ on $\mathbb{R}$. Using σ, this induces a smooth vector field on the circle minus the North Pole. Can it be extended to a smooth vector field on all of S^1?

2a) A smooth map $F : M \to N$ is a *submersion* if...

b) Let M be a compact, smooth 3-manifold. Prove that there is no submersion $F : M \to \mathbb{R}^3$.

3) Consider D the open unit disk in $\mathbb{R}^2$ with Riemannian metric

$$g = \left(\frac{2}{1 + x^2 + y^2}\right)^2 dx \otimes dx + \left(\frac{2}{1 + x^2 + y^2}\right)^2 dy \otimes dy$$

a) Write down an (oriented) orthonormal frame (E_1, E_2) for D with respect to this metric.

b) Write down the associated dual coframe $(\varepsilon^1, \varepsilon^2)$.

c) Compute $\varepsilon^1 \wedge \varepsilon^2$. Is this the Riemannian volume form (that is, does it agree with the volume formula $\sqrt{\det(g_{ij})}\, dx \wedge dy$)?

d) Compute the volume (area?) of D with respect to this metric.

e) What have you computed?

4) Suppose that f_0 and f_1 are smoothly homotopic maps from X to Y and that X is a